Automation Systems
Lecture 2 - Mathematical Models of Dynamical Systems

Jakub Mozaryn

Institute of Automatic Control and Robotics, Department of Mechatronics, WUT

Warszawa, 2016
Real processes, and thus control systems, have **nonlinear** properties:

- turbulences,
- multiple stable states,
- hysteresis,
- energy losses due to friction.

In practice, to simplify the mathematical description, there is carried **linearization**, enabling the formulation of the approximate description of a linear phenomenon, in vicinity of the operating point (this point corresponds to the most nominal or averaged operating conditions of the system).

- description of the phenomenon in the form of differential equations,
- linearization,
- operational calculus: differential equations $\rightarrow$ algebraic equations.
6 step approach to modelling

- **STEP 1**: Define system and its components.
- **STEP 2**: Formulate the mathematical model and fundamental necessary assumptions based on basic principles.
- **STEP 3**: Obtain differential equations representing the mathematical model.
- **STEP 4**: Solve equations for the desired output variables.
- **STEP 5**: Examine the assumptions and solutions.
- **STEP 6**: In necessary, reanalyze and redesign the system.
The basic forms of mathematical description of the operation of the (system) are:

- **Equations of Motion**: equations of system dynamics in form of differential equations.

- **Transfer function**.

- **State Space Equations**.

In the case of dynamical system (process) with one input signal $x(t)$ and one output signal $y(t)$ 
**equation of motion** describes the **relationship** between the output signal $y(t)$ and the input signal $x(t)$ and has a following form:

$$y(t) = f(x(t))$$ (1)
Description of linear models / systems

**Principle of superposition:**

\[ f(x_1 + x_2) = f(x_1) + f(x_2), \text{ and } f(0) = 0 \]  

(2)

Space of solutions of the equation that satisfies (2) is a **linear space**.

**Homogeneity (implies scale invariance):**

Function \( f(x, y) \) is said to be homogeneous of degree \( k \) if

\[ f(\beta x, \beta y) = \beta^k f(x, y), \text{ and } f(0) = 0 \]  

(3)

. where: \( \beta \) - constant coefficient.

**Linear system**

Homogenous system, which preserve the principle of superposition.

**Nonlinear system**

The system, which does not preserve the **principle of superposition** and/or is not **homogenous**.
Description of linear models / systems

General form of linear system differential equation:

\[ a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \cdots + b_0 x \] (4)

where: \( y \) - output signal, \( x \) - input signal, \( a_i, b_i \) - constant coefficients.
Static characteristic

Static characteristic $f_s$ describes the dependence of the output signal $y$ of the system from the input signal $x$ in **steady state**.

Steady state

Steady state is a state in which **all derivatives of the input signal and output signal are equal to zero**.

**Figure 1**: Static characteristics of linear system.
Creation of linear description of the system, based on nonlinear description of this system is called **linearization**.

Linearization of nonlinear description in the form of nonlinear algebraic equations is called **static linearization**. (There are no derivatives)

Linearization of nonlinear description in the form of nonlinear differential equations is called the **dynamic linearization**.

Methods of static linearization

- **secant method**: obtain the best correspondence between linear and non-linear description of a system in the specified range of changes of the independent variable (input).

- **tangent method**: obtain the best correspondence between linear and non-linear description of a system for a given value of the independent variable (input), and hence a particular value of the dependent variable (output).
Static linearization

Figure 2: Static linearization; a) secant method, b) tangent method.

In automation, there is considered the behavior of systems in a vicinity of a specific operating point. Therefore in practical applications tangent linearization method is much useful.
Process of linearization using **tangent method** involves:

- replacement of the curve representing nonlinear relationship $y = f(x)$ with its tangent at operating point,
- transfer the origin to operating point,
- replacement in mathematical model absolute variables $x$ and $y$ with deviations of these variables from operating point - incremental variable $\Delta x$ and $\Delta y$.

Static characteristic obtained using linearized equation, in terms of the specified operating point, is a linear function. It can be also obtained by linearization of real characteristic in terms of the same operating point.
Dynamic linearization

An example of differential equation, which describes linear relationship between functions \( x(t), y(t) \) and their derivatives.

\[
F[y(t), \dot{y}(t), \ddot{y}(t), \ldots, y^{(n)}(t), x, \dot{x}(t), \ddot{x}(t), \ldots, x^{(m)}(t)] = 0 \quad (5)
\]

During **dynamic linearization**, functions \( x(t), y(t) \) and their derivatives are treated analogously to variables of **implicit function**.

\[
\sum_{i=0}^{n} \left\{ \left[ \frac{\partial F}{\partial y^{(i)}} \right] y^{(i)}_0 \Delta y^{(i)} \right\} + \sum_{j=0}^{m} \left\{ \left[ \frac{\partial F}{\partial x^{(j)}} \right] x^{(j)}_0 \Delta x^{(j)} \right\} = 0 \quad (6)
\]

where:

\[
\Delta y = y(t) - y_0, \quad \Delta \dot{y} = \frac{d \Delta y}{dt}, \ldots, \quad \Delta y^{(n)} = \frac{d^n \Delta y}{dt^n}
\]

\[
\Delta x = y(t) - x_0, \quad \Delta \dot{x} = \frac{d \Delta x}{dt}, \ldots, \quad \Delta x^{(m)} = \frac{d^m \Delta x}{dt^m}
\]
Dynamic linearization - example

Take an example - non-homogenous function

\[ y = mx + b \]  \hspace{1cm} (7)

The normal operating point - \( \{x_0, y_0\}\), \( y_0 = f(x_0) \)

Taylor series expansion about the operating point

\[ y = f(x) = f(x_0) + \frac{df}{dx}|_{x=x_0} \frac{(x - x_0)}{1!} + \frac{d^2f}{dx^2}|_{x=x_0} \frac{(x - x_0)^2}{2!} + \ldots \]  \hspace{1cm} (8)

the slope (first derivative) over the operating point is good approximation of the curve over small range.

Therefore

\[ y = f(x_0) + \frac{df}{dx}|_{x=x_0} (x - x_0) = y_0 + m(x - x_0) \]  \hspace{1cm} (9)

and finally

\[ y - y_0 = m(x - x_0) \rightarrow \Delta y = m\Delta x \]  \hspace{1cm} (10)
Laplace transform

Replacing differential equation with transfer function (algebraic equation) needs transition from the time domain \( t \) to the complex plane \( S \).

\[
f(t) \Leftrightarrow f(s), \text{ where } s = c + j\omega \tag{11}\]

where: \( c \) - real part coefficient, \( \omega \) - conjugate part coefficient.

Laplace transform

\[
f(s) = L[f(t)] = \int_0^\infty f(t)e^{-st}dt \tag{12}\]

Inverse Laplace transform - Riemann Mellin integral

\[
f(t) = L^{-1}[f(s)] = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} F(s)e^{st}ds \tag{13}\]
Laplace transform is used for an analysis of control systems. As a tool for graphical analysis, complex plane $S$ is used, where multiplication by $s$ has the effect of differentiation and division by $s$ has the effect of integration. Analysis of complex roots of a linear equation, may disclose information about the frequency characteristics and the stability of the system.

To determine the function’s transform the following conditions must be meet:

- $f(t)$ has a finite value in any finite interval,
- $f(t)$ has a derivative $\frac{df(t)}{dt}$ in any finite interval,
- there exists a set of real numbers $X$ for which the integral $\int_{0}^{\infty} e^{-ct}$ is absolutely convergent.
Laplace transform of the linear systems

Linear system is described by following differential equation

\[ a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \cdots + b_0 x \]  \hspace{1cm} (14)

using *Laplace* transformation

\[ L \left[ \frac{d^n y}{dt^n} \right] = s^n y(s) - s^{n-1} y(0^+) - \cdots - y^{n-1}(0^+) \]  \hspace{1cm} (15)

and assuming that initial conditions are zero, one obtains

\[ L \left[ \frac{d^n y}{dt^n} \right] = s^n y(s) \]  \hspace{1cm} (16)

*Laplace* transform linear system with zero initial conditions take the following form

\[ y(s)(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0) = x(s)(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0) \]  \hspace{1cm} (17)
Transfer function

For continuous-time input signal $x(t)$ and output $y(t)$, the transfer function $G(s)$ is the linear mapping of the Laplace transform of the input, $X(s) = L[x(t)]$, to the Laplace transform of the output $Y(s) = L[y(t)]$ at zero initial conditions:

$$y(s)(a_ns^n + a_{n-1}s^{n-1} + \cdots + a_0) = x(s)(b_ms^m + b_{m-1}s^{m-1} + \cdots + b_0)$$  \hspace{1cm} (18)

$$G(s) = \frac{y(s)}{x(s)} = \frac{b_ms^m + b_{m-1}s^{m-1} + \cdots + b_0}{a_ns^n + a_{n-1}s^{n-1} + \cdots + a_0}$$ \hspace{1cm} (19)

Numerator

$$M(s) = b_ms^m + b_{m-1}s^{m-1} + \cdots + b_0$$ \hspace{1cm} (20)

Denominator - characteristic equation

$$N(s) = a_ns^n + a_{n-1}s^{n-1} + \cdots + a_0$$ \hspace{1cm} (21)
Determination of static characteristic from transfer function

\[ x_0 = \lim_{t \to \infty} x(t), \quad y_0 = \lim_{t \to \infty} y(t), \quad (22) \]

using the **final value theorem**

\[ y_0 = \lim_{t \to \infty} y(t) = \lim_{s \to 0} s y(s) = \lim_{s \to 0} sG(s)x(s) \quad (23) \]

\[ x_0 = \text{const} \Rightarrow x(s) = \frac{1}{s} x_0 \quad (24) \]

\[ \frac{y_0}{x_0} = \lim_{s \to 0} G(s) \quad (25) \]

finally

\[ y_0 = \frac{b_0}{a_0} x_0 \quad (26) \]
Typical input signals

Unit step (Heaveside function)

\[ x(t) = \begin{cases} 
1(t) & \text{for } t \geq 0 \\
0 & \text{for } t < 0 
\end{cases} \]

\[ x(s) = \frac{1}{s} \]

Step with constant value

\[ x(t) = \begin{cases} 
x_{st}1(t) & \text{for } t \geq 0 \\
0 & \text{for } t < 0 
\end{cases} \]

\[ x(s) = x_{st} \frac{1}{s} \]

Impulse - Dirac delta function

\[ x(t) = \delta(t) = \begin{cases} 
0 & \text{for } t \neq 0 \\
\infty & \text{for } t = 0 
\end{cases} \]

\[ x(s) = 1 \]

Ramp

\[ x(t) = at \]

\[ x(s) = \frac{a}{s^2} \]
Changes of output signal $y(t)$ as a response to specific change of input signal $x(t)$

Figure 3: Example of transient of the dynamical system
Methods for determining the transient response of the system

\[ a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \cdots + b_0 x \]  

(27)

Classic:

- Assumption of the initial conditions \( x(0), y(0) \).
- Solution of differential equations.

Using transfer function:

\[ f(t) = L^{-1}[y(s)] = L^{-1}[G(s)x(s)] \]

(28)

To perform Laplace transformation and its reverse, which are the basic operations of the transfer function calculus, its often sufficient to know the basic properties of **transfer functions and tables of transfer functions**.